① Q92 p.793 (Thomas's Calculus
George B. Thomas, Jr., 12 edision)

Let $f(x,y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise} \end{cases}$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist,
but f is not differentiable at $(0,0)$

② Math 2010 D, Problem set 6 : Q6.

Let $$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

(a) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$

(b) Prove that f is differentiable at $(0,0)$.

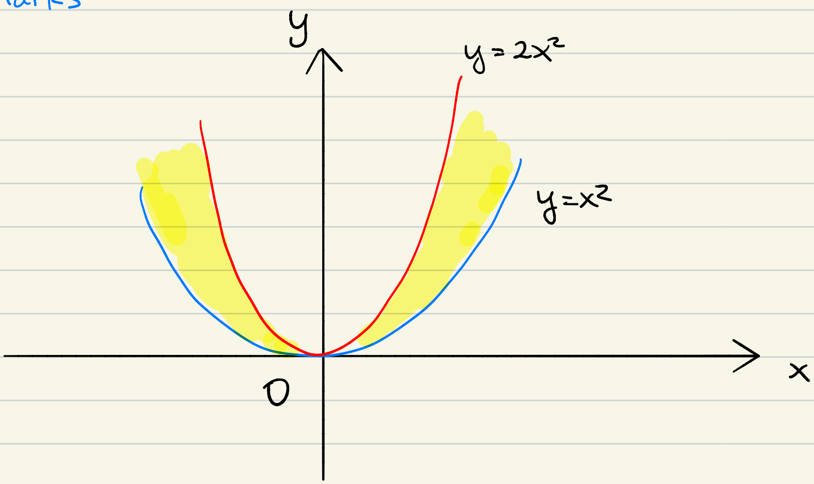
③ If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ is a linear map,
then the total derivative $Df(\vec{a})$ at the point $\vec{a}$
is f. $[\ Df(a) = f\ ]$

a $1 \times n$ matrix   a mapping : $\mathbb{R}^n \to \mathbb{R}^1$

Solution

1. $$f(x,y) = \begin{cases} 0 & \text{if } x^2 < y < 2x^2 \\ 1 & \text{otherwise} \end{cases}$$

Remarks

① 

The function $f(x,y)$ is discontinuous at $(0,0)$

Consider the curve $\gamma(t) = (t, \frac{3}{2}t^2) = (x,y)$
(in xy - coordinate)

Checked that $\gamma(t)$ lies inside the shaded region when $t \neq 0$.

$\therefore f(\gamma(t)) = 0$ if $t \neq 0$

Note that

(i) $\gamma(t) \to 0$ as $t \to 0$

(ii) $f(\gamma(t)) = 0$ for $t \neq 0$

(iii) $f(0,y) = 1$ for $y \neq 0$

Conclusion : f has non-removable discontinuity at $(0,0)$ $\left(\lim_{t\to 0} f(\gamma(t)) = 0 , \lim_{y \to 0} f(0,y) = 1 \right)$

$\therefore$ f is not differentiable at $(0,0)$

Moreover,

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x}$$

$$\left(\because \begin{array}{l} x^2 < 0 \text{ is false,} \\ f(x,0) = 1 \end{array} \right) = \lim_{x \to 0} \frac{1-1}{x} = 0$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y}$$

$$\left(\because \begin{array}{l} 0 < y < 0 \text{ is false,} \\ f(0,y) = 1 \end{array} \right) = \lim_{y \to 0} \frac{1-1}{y} = 0$$

$\therefore$ $f_x(0,0) = f_y(0,0) = 0$ exists

Remark : Good exercise.

① Indeed, $\left[\frac{\partial f}{\partial \vec{u}} = 0 \text{ for any unit vector } \vec{u}. \right]$

This is an example that every directional derivative exists, but the function is not differentiable.

② Warning : You cannot deduce that $\frac{\partial f}{\partial \vec{u}} = 0$ due to $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$, when f is <u>NOT</u> differentiable at that point.

Example : Lecture notes 6 p. 79

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

($\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$, but

$$\frac{\partial f}{\partial \vec{u}}(0,0) = \lim_{r \to 0} \frac{f(r\cos\theta, r\sin\theta)}{r} \quad (\vec{u} = (\cos\theta, \sin\theta))$$

$$= \lim_{r \to 0} \frac{r^2\cos\theta\sin\theta}{r^3} = \lim_{r \to 0} \frac{\cos\theta\sin\theta}{r}$$

even does not exist when $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.)

2.

$$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

(a) $$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

(b) If f is differentiable at $(0,0)$, then

$$Df(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right)$$

$$= (0, 0)$$

Let $\vec{h} = (h_1, h_2)$, let $\vec{T} = (0,0)$

$$\frac{1}{|\vec{h}|}\left[f(\vec{h}) - f(0,0) - \vec{T}\cdot\vec{h} \right]$$

$$= \begin{cases} \frac{1}{\sqrt{h_1^2 + h_2^2}} \left[h_1^3 \sin\frac{1}{h_1^2} + h_2^3 \sin\frac{1}{h_2^2} \right] & \text{if } h_1 h_2 \neq 0 \\ 0 & \text{if } h_1 h_2 = 0 \end{cases}$$

Put $h_1 = r\cos\theta$, $h_2 = r\sin\theta$

$$= \begin{cases} r^2\left(\cos^3\theta \sin\frac{1}{h_1^2} + \sin^3\theta \sin\frac{1}{h_2^2}\right) & \text{if } h_1h_2 \neq 0 \\ 0 & \text{if } h_1h_2 = 0 \end{cases}$$

Note $|\vec{h}| = r$

Sandwich thm

$$\Rightarrow \lim_{|\vec{h}|\to 0} \frac{1}{|\vec{h}|}\left[f(\vec{h}) - f(\vec{0}) - \vec{T}\cdot\vec{h}\right] = 0$$

$\therefore$ f is differentiable at $(0,0)$ with total derivative $Df(0,0) = \langle 0, 0 \rangle$

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3. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be a linear map if

for every scalar $\alpha \in \mathbb{R}$, vector $\vec{x}, \vec{y} \in \mathbb{R}^n$,

we have $f(\alpha\vec{x} + \vec{y}) = \alpha f(\vec{x}) + f(\vec{y})$

Thm: Every linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is just a matrix multiplication mapping.

More precisely, you can always find a

$m \times n$ matrix M so that

$$f(\vec{v}) = M\vec{v} \quad \text{for every } v \in \mathbb{R}^n$$

Matrix multiplication

Remark :

① The choice of matrix M depends on the function f

② Once your M is chosen,

$$f(\vec{v}) = M\vec{v} \quad \text{for every } v \in \mathbb{R}^n$$

③
$$M = \begin{pmatrix} | & | & & | \\ f(\vec{e}_1) & f(\vec{e}_2) & \cdots & f(\vec{e}_n) \\ | & | & & | \end{pmatrix}$$

④
$$e_i \in \mathbb{R}^n, \quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{ith position}$$

$$f(\vec{e}_i) \in \mathbb{R}^m \quad (\because f : \mathbb{R}^n \longrightarrow \mathbb{R}^m)$$

You can see easily that M is really a $m \times n$ matrix

⑤ Verification : Let $\vec{v} \in \mathbb{R}^n$, there are some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, so that

$$\vec{v} = \alpha_1\vec{e_1} + \alpha_2\vec{e_2} + \cdots + \alpha_n\vec{e_n}$$

$$\begin{aligned}
f(\vec{v}) &= f(\alpha_1\vec{e_1} + \alpha_2\vec{e_2} + \cdots + \alpha_n\vec{e_n}) \\
&= \alpha_1 f(\vec{e_1}) + \alpha_2 f(\vec{e_2}) + \cdots + \alpha_n f(\vec{e_n}) \\
&= \begin{pmatrix} | & | & & | \\ f(\vec{e_1}) & f(\vec{e_2}) & \cdots & f(\vec{e_n}) \\ | & | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \\
&= M(\alpha_1\vec{e_1} + \alpha_2\vec{e_2} + \cdots + \alpha_n\vec{e_n}) \\
&= M\vec{v}.
\end{aligned}$$

$\cancel{\times}$

Back to our assertion, more precisely,

if M_f is the matrix so that $f(\vec{v}) = M_f\vec{v}$,

then $Df(\vec{a}) = M_f$ for every $a \in \mathbb{R}^n$

Proof :

$$\begin{aligned}
& f(\vec{a}+\vec{h}) - f(\vec{a}) - M_f \odot \vec{h} \\
&= f(\vec{a}+\vec{h}) - f(\vec{a}) - f(\vec{h}) \\
&= 0 \qquad \forall \vec{h} \in \mathbb{R}^n
\end{aligned}$$

You may regard $\vec{h}$ as a $n\times1$ matrix, then this is a matrix multi. only.